Title: Introduction to Limits

Grade Ranges:
___K-4
___5-8
_X_9-12

Subject Tag:
Math: Pre-calculus
Math: Problem Solving
Math: Algebra

Synopsis:
Students will be introduced to the concept of a limit through the exploration of numeric limits using sequences and functions. Spreadsheets and graphing calculators will be used throughout the investigation. This lesson also uses a frog who mathematically tries to reach the love of his life.

Keywords:
limits, convergence, divergence, sequences, graphic calculators, spreadsheets, harmonic sequences

Body:
1. Provide the following sequence of numbers and the questions that follow to your students. This chart is included in Handout 1.

   Consider the sequences of numbers:

n	1	2	3	4	5	6
S_1	2	4	6	8	10	12
S_2	25	23	20	16	11	5
S_3	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
S_4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

   What is the next term in each sequence? What about the 100^{th} term?
   If each pattern continued forever, what happens to the terms of the sequence?

<u>Notes for teachers:</u> Using a graphing calculator or a spreadsheet, create a list of the terms of each sequence. By using the seq() command on TI graphing calculators, you can fill an entire list with the positive integers to a value of your choice. On a spreadsheet, the fill-down command allows you to quickly copy a formula into an entire column.

For example:

n	Seq 1	Seq 2	Seq 3	Seq 4
1	2	25	1	1
2	4	23	0.5	0.5
3	6	20	0.333333	0.25
4	8	16	0.25	0.125
5	10	11	0.2	0.0625
6	12	5	0.166667	0.03125
7	14	-2	0.142857	0.015625
8	16	-10	0.125	0.007813

In the spreadsheet, "1" was entered in A2 and the formula "=1+A2" was entered in A3 and filled down the column. Seq 1 was created using the formula "=2*A2" and filling down. Seq 2 was created by entering "25" in C2 and then using the formula "=C2-A3" and filling down. Seq 3 was created by using the formula "=1/A2." Seq 4 was created by using the formula "=(1/2)^(A2-1)."

Students should quickly notice the behavior of the first two sequences — one increases and the other decreases. Also, they may quickly realize that the 3rd and 4th sequences approach zero. Ask students if they think the sequences will ever reach zero.

2. Present the story of Froggy and Froggy's revenge to your students:

Froggy is a lonely, teenage frog who has his heart set on this incredible smart, attractive, athletic frog, Wanda. One nice, sunny day on the pond, Froggy gets the nerve up to ask Wanda if he can come on over to her pad (lilypad) for a while. Wanda is not all that interested, but because Froggy persists, she says he can come over on one condition. The excited Froggy leaps at the chance and eagerly awaits her condition. Wanda says, "Each day, you can move your lilypad half the distance to my lilypad. When you reach me, you can stay as long as you like." Froggy knows that her pad is only 2 units away, so he figures it will only be a matter of time before he is next to his frog. How long before Froggy reaches his beloved? Keep in mind, these frogs are unique in that not only do they use math in romance, they are also point-sized frogs.

<u>For Discussion with students</u>:
Which one of the sequences above (on your handout), corresponds to this situation? From this sequence, create a sequence that represents Froggy's total distance traveled at that time.

A spreadsheet of values is listed here:

n	Daily distance	Total Distance
1	1	1
2	0.5	1.5
3	0.25	1.75
4	0.125	1.875
5	0.0625	1.9375
6	0.03125	1.96875

7	0.015625	1.984375
8	0.0078125	1.9921875
9	0.00390625	1.99609375
10	0.00195313	1.998046875

What happens to Froggy's total distance?
Does he ever actually reach his sweet Wanda?

Once students figure out that Froggy will never actually reach Wanda, introduce part two of the story:

Froggy's revenge
Dejected, Froggy returns to his original position and begins to study some mathematics. He learns of a different sequence of numbers, $\frac{1}{2},\frac{1}{3},\frac{1}{4}$*... a Harmonic sequence, and he decides to pursue his one and only Wanda armed with a little more knowledge. Froggy proposes to Wanda that she allow him to come half of the total way (one unit) on the first day, one-third of the total way on the second day, one-fourth of the total way on the third day, and so on. Wanda ponders that sequence of numbers and concludes that since the sequence of those fractions approaches zero, Froggy's attempt to win her affection will again fail. She agrees. Much to her surprise and to Froggy's pleasure, he reaches her pad in a matter of days, and they live happily ever after.*

<u>For Students, independently:</u> Create a spreadsheet or a list in your graphing calculator to model this sequence of partial sums.

3. Student investigation: (Note: this section appears in Handout 2.)

Now consider the graph of the function: $y = \frac{x+3}{x+4}$.

a. Are there any values of x for which we cannot calculate a value of y? Why?
b. Enter this function in Y_1 in your graphing calculator.
c. Using the table set, begin your table at x=-5 and set the ΔTable to .05. What happens to the value of y as x changes from –5 to –4? What happens to the value of y as x changes from –3 to –4?
d. Explain these behaviors in your own words.
e. How does the graph reflect this behavior?

Now consider the function: $y = \frac{x-3}{x^2-x-6}$.

a. Are there any values of x for which we cannot calculate a value of y? Why?
b. Enter this function in Y_1 in your graphing calculator.

c. Using the table set, begin your table at x=2 and set the ΔTable to 0.1. What happens to the value of y as x changes from 2 to 3? What happens to the value of y as x changes from 4 to 3?
d. Explain these behaviors in your own words.
e. How does the graph reflect this behavior?
f. Is there another graph that would be virtually identical to this graph? In what ways would it be different?

Now consider the function: $y = \lfloor x \rfloor$ (the greatest integer function).

a. Enter this function in Y_1 in your graphing calculator.
b. Using the table set, begin your table at x=1 and set the ΔTable to 0.1. What happens to the value of y as x changes from 1 to 2? What happens to the value of y as x changes from 2 to 1?
c. Explain these behaviors in your own words.
d. How does the graph reflect this behavior?

Objective:
To give students a concrete foundation for their conceptual understanding of limits

Standards:
NY: 1: Students use mathematical reasoning to analyze mathematical situations, make conjectures, gather evidence, and construct an argument. **2**: Students use number sense and numeration to develop an understanding of the multiple uses of numbers in the real world, the use of numbers to communicate mathematically, and the use of numbers in the development of mathematical ideas. **3**: Students use mathematical modeling/multiple representation to provide a means of presenting, interpreting, communicating, and connecting mathematical information and relationships.

NYC: M3d. Works with rates of many kinds expressed numerically, symbolically, and graphically. **M3g.** Uses arithmetic sequences and geometric sequences and their sums, and sees these as the discrete forms of linear and exponential functions, respectively. **M3i.** Represents functional relationships in formulas, tables and graphs, and translates between pairs of these. **M3k.** Makes predictions by interpolating and extrapolating from given data or a given graph. **M3n.** Uses technology such as graphics calculators to represent and analyze functions and their graphs.

CT: 8 . Patterns. Students will discover, analyze, describe, extend and create patterns, and use patterns to describe mathematical and other real-world phenomena. **9 .** Algebra and Functions. Students will use algebraic skills and concepts, including functions, to describe real-world phenomena symbolically and graphically, and to model quantitative change.

NJ: 4.2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression. **4.3.** All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas. **4.4.** All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers. **4.5.** All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power. **4.11.** All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.

Prerequisite Skills:
1. A basic knowledge of sequences
2. A mastery of the basic functions studied in Pre-calculus
3. A basic knowledge of Geometry

Time Required:
one to two class days

Technology and Materials Needed:
1. Graphing calculator with a list of features (a spreadsheet program can be substituted in some explorations
2. The ability to demonstrate/present to the class using Geometer's Sketchpad

Recommended Lesson Plan Review Date:

Review Comments:

Name:
Date:

Introduction to Limits

Consider the sequences of numbers:

n	1	2	3	4	5	6
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S_4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

1. What is the next term in each sequence?

   S_1
   S_2
   S_3
   S_4

2. What about the 100^{th} term?

   S_1
   S_2
   S_3
   S_4

3. If each pattern continued forever, what happens to the terms of the sequence?

   S_1

   S_2

   S_3

   S_4

Name:
Date:

Introduction to Limits, Part 2

1. Consider the graph of the function: $y = \frac{x+3}{x+4}$.

a. Are there any values of x for which we cannot calculate a value of y?

   Why?

b. Enter this function in Y_1 in your graphing calculator.
c. Using the table set, begin your table at x=-5 and set the ΔTable to .05.
   What happens to the value of y as x changes from –5 to –4?

   What happens to the value of y as x changes from –3 to –4?

d. Explain these behaviors in your own words.

e. How does the graph reflect this behavior?

2. Now consider the function: $y = \frac{x-3}{x^2 - x - 6}$.

a. Are there any values of x for which we cannot calculate a value of y?

   Why?

b. Enter this function in Y_1 in your graphing calculator.
c. Using the table set, begin your table at x=2 and set the ΔTable to 0.1.
   What happens to the value of y as x changes from 2 to 3?

   What happens to the value of y as x changes from 4 to 3?

d. Explain these behaviors in your own words.

e. How does the graph reflect this behavior?

f. Is there another graph that would be virtually identical to this graph?
   In what ways would it be different?

3. Now consider the function: $y = \lfloor x \rfloor$ (the greatest integer function).

a. Enter this function in Y_1 in your graphing calculator.
b. Using the table set, begin your table at x=1 and set the ΔTable to 0.1.
   What happens to the value of y as x changes from 1 to 2?

   What happens to the value of y as x changes from 2 to 1?

c. Explain these behaviors in your own words.

d. How does the graph reflect this behavior?