Title: Understanding Fundamental Trigonometric Identities

Grade Ranges:
___K-4
___5-8
_X_9-12

Subject Tag:
Math: Pre-Calculus

Synopsis:
In this lesson, students will use a diagram of a tangent drawn to the unit circle, and apply the basic definitions of the trigonometric ratios along with some geometric properties to: 1) understand why tangent, cotangent, cosecant, and secant are named as such, 2) verify the reciprocal trigonometric identities, 3) verify the Pythagorean trigonometric identities, and 4) verify the co function trigonometric identities.

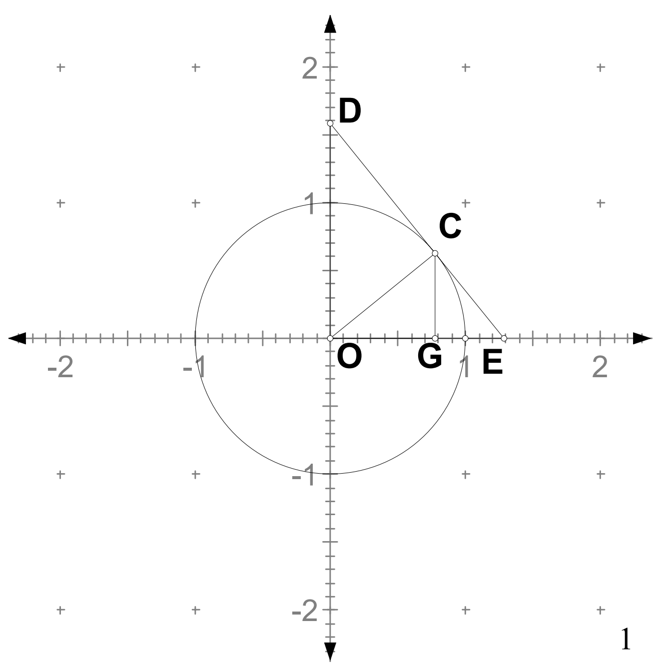
Keywords:
sine, cosine, tangent, cotangent, secant, cosecant, trigonometry, trigonometric identities, Pythagorean identities, co function identities, reciprocal identities

Body:

1. Introduce the lesson by reviewing the following geometric properties:
    a. The radius drawn to a tangent is perpendicular to the tangent at the point of tangency.
    b. If the altitude is drawn to the hypotenuse in a right triangle, three similar triangles are formed.
    c. In similar triangles, the ratios of corresponding side lengths are equal, and corresponding angles are congruent.
    d. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs of the triangle (The Pythagorean Theorem).

2. Have students construct a unit circle and draw a tangent segment from a point on the x-axis to a point on the y-axis. Draw a radius to the point of tangency. Drop a perpendicular to the x-axis from that point of tangency. The diagram should look like the one to the right (Figure 1) . Note to Cablevision: this image is in the PDF titled CBV.86.E.M.IM1.D1.pdf. Note: this diagram can be constructed easily with a dynamic piece of software like Geometer's Sketchpad. In doing so, the

Figure 1.



verification of these identities can be dynamically illustrated. Also, additional extensions are made possible and will be discussed later in this lesson plan.

To construct this diagram on Sketchpad:

(Note: These instructions for students are included in handout 1.)

a. Under the Graph menu, select Show grid.
b. Using the circle tool, draw a circle centered at the origin and passing through the unit point (1,0). To make the circle "attached" to the unit point, begin your circle by clicking on the center and dragging; un-click on the unit point.
c. Construct a free point on the circle.
d. Draw a radius from the center to the free point.
e. Select the free point on the circle and the radius and choose Construct…Perpendicular line.
f. After constructing points where this tangent line intersects each axis, hide the line. Now, draw segments from the point of tangency to the points on each axis. You should also draw segments from the origin to each of these points where the tangent intersects the x- and y-axes.
g. Select the point of tangency and the x-axis and choose Construct …Perpendicular line.
h. Again, construct the point of intersection of the perpendicular and the x-axis. Hide the perpendicular. Draw a segment from the point of tangency to the x-axis and from the origin to this point on the x-axis.

3. Use the geometric properties listed in (1), the diagram constructed in (2), and the definitions of trigonometric ratios in a right triangle, determine which segments in the diagram have a length equal to each of the six trigonometric ratios for the angle whose terminal side passes through the point of tangency — angle GOC. [Note: Henceforth, this angle will be referred to as "our angle."] Depending on the level of the student(s), this could be done completely independently or completely dependently or some combination of the two. For a good pre-calculus class:
   a. Review the definitions of the six trigonometric ratios in a right triangle.
   b. Begin by showing students which segment is equal in length to the sine of our angle — reminding them that the radius of the unit circle is 1 so the opposite leg divided by the hypotenuse is equal to the opposite leg. Now, let them try to determine which segment is equal to the cosine of our angle.
   c. Have students identify which other angles in the diagram are equal to our angle.
   d. Working in pairs or triples, have students work to find a segment whose length is equal to the tangent of our angle. *Hint: look for a triangle that contains our angle and the adjacent leg is equal to 1.* Have students explain their reasoning to the class.
   e. Challenge students to determine the segments corresponding to the other three trigonometric ratios. This could be completed for homework. The final labels should look like this: (Figure 2) Note to Cablevision: this image is in

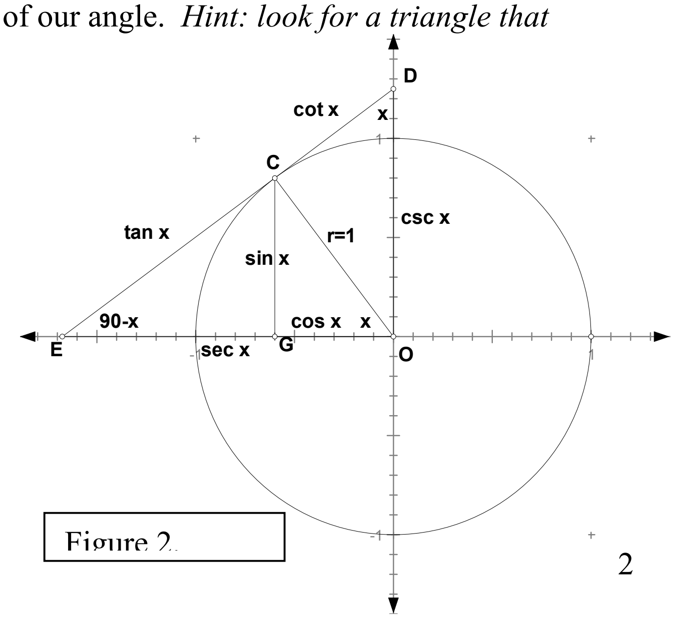


Figure 2.

the PDF titled CBV.86.E.M.IM2.D1.pdf.

f. Have students explain why the names of the trigonometric ratios are appropriate.

4. *Sketchpad Extension:* If you have constructed this diagram with Sketchpad, you can dynamically illustrate these properties. (Note: this first extension's instructions are also included in handout 1.)
   a. Measure from the origin to the unit point and adjust the size of a unit to equal 1.00 cm or inches (depending on the length unit preference).
   b. Change the labels of each of the segments so that it is easy to see which segments are equal to which trigonometric ratios.
   c. Measure the angle determined by the unit point, the origin, and the point of tangency by selecting the points in that order and choosing Measure…Angle.
   d. Measure the length of each of the segments identified in step (3) by selecting the segment and choosing Measure…Length.
   e. Double-click on the angle measure. The calculator will appear. From the function list, choose sin() and from the value list choose the angle. Click OK.
   f. Students should see that the length of the segment is equal to the sine of the angle. If the point of tangency is moved, the two remain equal.
   g. Complete a similar calculation for each trigonometric ratio and its corresponding length.

5. *Another Sketchpad Extension:* Move the free point around the circle and ask students to describe in their own words what is happening to the length of the segment that corresponds to the sine or cosine of the angle. Having students write their response to this demonstration instead of asking for volunteers will force all students to verbalize their observations. Now ask a similar question for the lengths of the tangent or cotangent and for the secant or cosecant. You can ask students to do the remaining explanations for homework, which will help students develop a deeper understanding of the graphs of the trigonometric functions, if they have already been studied, or an intuitive understanding of the functions before they have been studied.

6. Working in pairs or triples, have students use the geometric properties and the definitions of the trigonometric ratios in a right triangle to explain or verify the following trigonometric identities: (Note: These identities are available in handout 2.)
   a. $\tan x = \frac{\sin x}{\cos x}$
   b. $\cot x = \frac{\cos x}{\sin x}$
   c. $\cot x = \tan(90 - x)$
   d. $\csc x = \sec(90 - x)$
   e. $\cos x = \sin(90 - x)$
   f. $\sin^2 x + \cos^2 x = 1$
   g. $\sec^2 x = \tan^2 x + 1$
   h. $\csc^2 x = \cot^2 x + 1$

7. *Graphing Calculator or Spreadsheet Extension.* (Note: these directions for students are available in handout 3.) Set your calculator to Degree Mode. Using the list editor

on a graphing calculator or on a spreadsheet, enter the values 0 to 90 in list 1 (or column 1). In list 2 (or column 2), enter the sin (L1). [Note: by moving the cursor to the name of the list at the top of L2 on the graphing calculator, sin(L1) can be entered and the entire list is filled with the corresponding value of the sine for the angle in L1. On the spreadsheet, put a formula in the first box, "=sin(A1)" and then drag that formula to fill down.] In L3, fill down the cos(L1). Take a few minutes to discuss what is happening to the values of the sine and cosine of the angle as the angle changes from 0 to 90. In L4, put in the sum of the squares of L2 and L3. The students can get a concrete understanding of the Pythagorean identity they just verified. Similar steps can also reinforce the other identities.

Related Links:
Trig Without Tears, by Oak Road Systems
http://oakroadsystems.com/math/trigsol.htm
http://oakroadsystems.com/math/trig10.htm
http://oakroadsystems.com/twt/index.htm
Here, students can find some alternative explanations of the trigonometric identities as well as helpful hints for memorizing them. Additionally, this site includes the derivation of the tangent/secant and cotangent/cosecant Pythagorean identities from the sine/cosine Pythagorean identity. In addition to being an exhaustive source for trigonometric identities, this site provides a thorough explanation (including many proofs and hints for memorization) of trigonometric topics studied in almost any high school course.

Features:
___Contains special education tips
___Quick Activity (less than 30 minutes; story starter)
___Requires Internet access for students to complete

Objectives:
The primary objectives of this lesson are to move students to a conceptual understanding of why these fundamental trigonometric identities are true rather than simply asking them to memorize them. Students gain familiarity with these identities and an appreciation for what it means for some statement to be a trigonometric identity. With the appropriate amount of practice following this exercise, students would be ready to apply these identities to verify other identities.

Standards:
NY: 1.2: Deductive and inductive reasoning are used to reach mathematical conclusions. **1.3:** Critical thinking skills are used in the solution of mathematical problems. **2.1:** Information technology is used to retrieve, process, and communicate information and as a tool to enhance learning. **3.1:** Students use mathematical reasoning to analyze mathematical situations, make conjectures, gather evidence, and construct an argument. **3.4:** Students use mathematical modeling/multiple representation to provide a means of presenting, interpreting, communicating, and connecting mathematical information and relationships.

NYC: M2b. Works with two- and three-dimensional figures and their properties, including polygons and circles, cubes and pyramids, and cylinders, cones, and spheres. **M2f.** Uses the Pythagorean Theorem in many types of situations, and works through more than one proof of this theorem. **M2g.** Works with similar triangles, and extends the ideas to include simple uses of the three basic trigonometric functions. **M2q.** Explores geometry using computer programs such as CAD software, Sketchpad programs, or LOGO. **M5a.** Formulation: The student participates in the formulation of problems. **M5c.** Conclusion: The student provides closure to the solution process through summary statements and general conclusions. **M6d.** Uses basic geometric terminology accurately, and deduces information about basic geometric figures in solving problems. **M6j.** Uses technology to create graphs or spreadsheets that contribute to the understanding of a problem.

CT: 6: Spatial Relationships and Geometry. Students will analyze and use spatial relationships and basic concepts of geometry to construct, draw, describe and compare geometric models and their transformations, and use geometric relationships and patterns to solve problems. **8:** Patterns. Students will discover, analyze, describe, extend and create patterns and use patterns to describe mathematical and other real-world phenomena.

NJ: 4.2: All Students Will Communicate Mathematically Through Written, Oral, Symbolic, And Visual Forms Of Expression. **4.4:** All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers. **4.5:** All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.

Prerequisite Skills:
1. Knowledge and familiarity of geometric properties
2. Knowledge and familiarity with the definitions of the six trigonometric ratios in a right triangle

Time Required:
Depending on the use and depth of the extensions, this investigation should take 2 to 3 class days.

Technology and Materials Needed:
1. Pencil and paper (graph paper optional)
2. Graphing Calculator or access to a spreadsheet (if extensions are used)
3. Access to Geometer's Sketchpad (if extensions are used)

Procedures:

Assessment Criteria:
1. Having students write their descriptions of the relationships or explanations of why a particular relationship holds true helps reveal how much each student understands about these concepts.
2. Use a traditional assignment of trigonometric identities to show the application of these identities.

Recommended Lesson Plan Review Date:

6 months

Review Comments:
Geometer's Sketchpad has just recently released an updated version 4.0. The steps for the construction of the figure using Sketchpad are for version 3.1 and probably will not need to be changed for two reasons; first, most schools may not have updated to that version, and second, the steps will most likely be the same.

(1)
Diagram (Figure 1) construction instructions for Sketchpad:

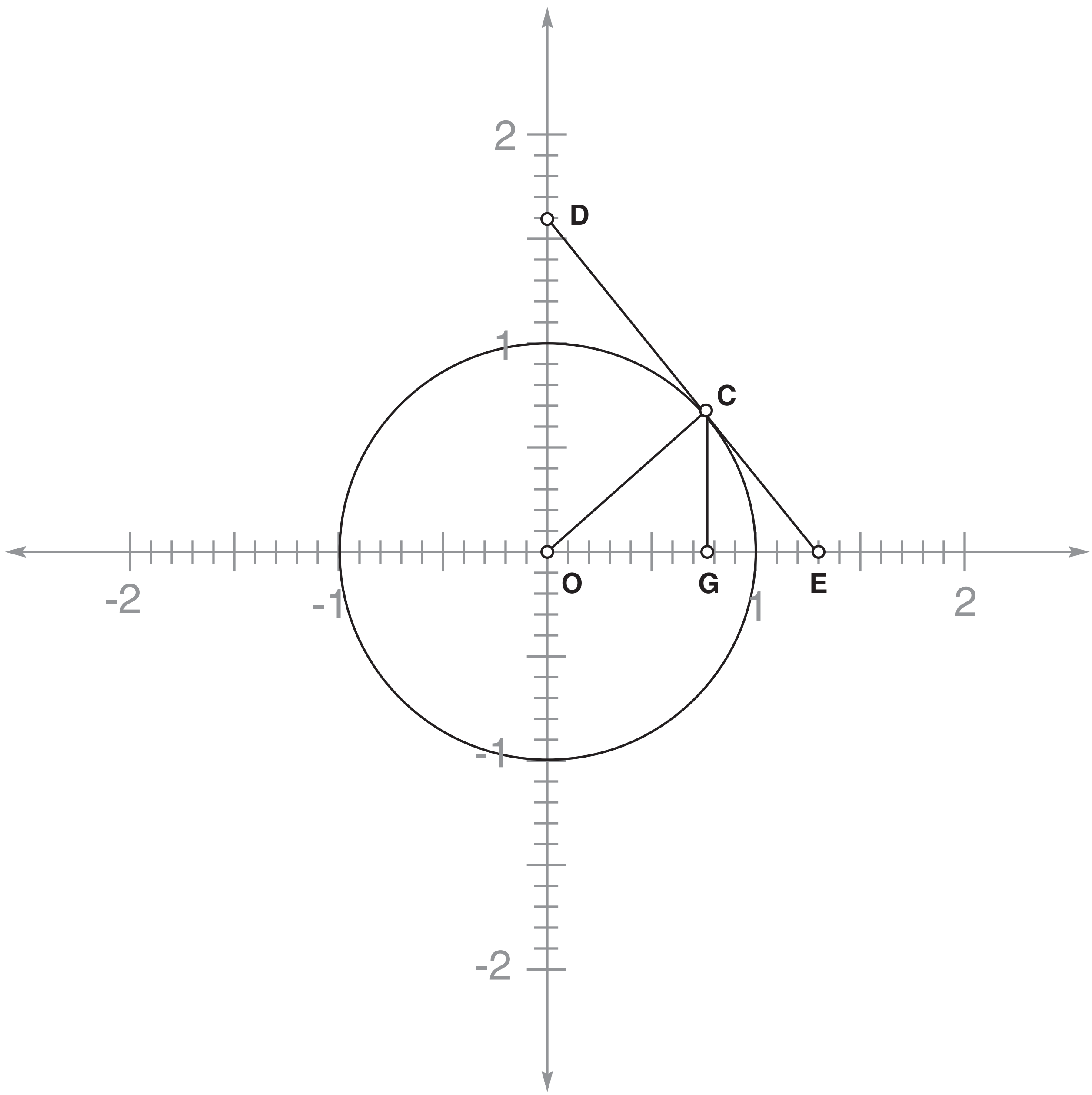
a. Under the Graph menu, select Show grid.
b. Using the circle tool, draw a circle centered at the origin and passing through the unit point (1,0). To make sure the circle is "attached" to the unit point, begin your circle by clicking on the center and dragging making sure you un-click on the unit point.
c. Construct a free point on the circle.
d. Draw a radius from the center to the free point.
e. Select the free point on the circle and the radius and choose Construct…Perpendicular line.
f. After constructing points where this tangent line intersects each axis, hide the line. Now, draw segments from the point of tangency to the points on each axis. You should also draw segments from the origin to each of these points where the tangent intersects the x- and y-axes.
g. Select the point of tangency and the x-axis and choose Construct…Perpendicular line.

Again, construct the point of intersection of the perpendicular and the x-axis. Hide the perpendicular. Draw a segment from the point of tangency to the x-axis and from the origin to this point on the x-axis.

Extension 1:

a. Measure from the origin to the unit point and adjust the size of a unit to equal 1.00 cm or inches (depending on the length unit preference).
b. Change the labels of each of the segments so that it is easy to see which segments are equal to which trigonometric ratios.
c. Measure the angle determined by the unit point, the origin, and the point of tangency by selecting the points <u>in that order</u> and choosing Measure…Angle.
d. Measure the length of each of the segments identified in step (3) by selecting the segment and choosing Measure…Length.
e. Double-click on the angle measure. The calculator will appear. From the function list, choose sin() and from the value list choose the angle. Click OK.
f. Students should see that the length of the segment is equal to the sine of the angle. If the point of tangency is moved, the two remain equal.
g. Complete a similar calculation for each trigonometric ratio and its corresponding length.

Figure 1



Understanding Fundamental Trigonometric Identities (3)

Graphing Calculator or Spreadsheet Extension

Calculator:

1. Set your calculator to Degree Mode.
2. Using the list editor on a graphing calculator or on a spreadsheet, enter the values 0 to 90 in list 1 (or column 1).
3. In list 2 (or column 2), enter the sin (L1). [Note: by moving the cursor to the name of the list at the top of L2 on the graphing calculator, sin(L1) can be entered and the entire list is filled with the corresponding value of the sine for the angle in L1.]

Spreadsheet:

1. On the spreadsheet, put a formula in the first box, "=sin(A1)" and then drag that formula to fill down.]
2. In L3, fill down the cos(L1). Take a few minutes to discuss what is happening to the values of the sine and cosine of the angle as the angle changes from 0 to 90.
3. In L4, put in the sum of the squares of L2 and L3.

Understanding Trigonometric Identities (2)

Names:
Date:

Working in pairs or triples, use the geometric properties and the definitions of the trigonometric ratios in a right triangle to *explain* or *verify* the following trigonometric identities:

a. $tan\, x = \dfrac{sin\, x}{cos\, x}$

b. $cot\, x = \dfrac{cos\, x}{sin\, x}$

c. $cot\, x = tan(90 - x)$

d. $csc\, x = sec(90 - x)$

e. $cos\,x = sin(90 - x)$

f. $sin^2 x + cos^2 x = 1$

g. $sec^2 x = tan^2 x + 1$

h. $csc^2 x = cot^2 x + 1$

Figure 2